Further Maths Revision Paper 6

This paper consists of 5 questions covering CP1, CP2, FP1 and FM1.
(AS Further Maths: Q1 and Q5)

1

Prove by induction that

$$\sum_{r=1}^{n} \frac{r2^r}{(r+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$$

Show true for $n=1$

$$\frac{1(2^1)}{(3)!} = \frac{1}{3} \qquad 1 - \frac{2^2}{3!} = 1 - \frac{2}{3} = \frac{1}{3}$$

Assume true for $n=k$

$$\sum \frac{k2^k}{(k+2)!} = 1 - \frac{2^{k+1}}{(k+2)!}$$

Show true for $n=k+1$

$$\sum \frac{k2^k}{(k+2)!} + \frac{(k+1)2^{(k+1)}}{(k+3)!} = 1 - \frac{2^{k+1}}{(k+2)!} + \frac{(k+1)2^{(k+1)}}{(k+3)!}$$

$$= 1 - \frac{(k+3)2^{k+1}}{(k+3)!} + \frac{(k+1)2^{(k+1)}}{(k+3)!}$$

$$= 1 - \frac{2^{k+1}(k+3-k-1)}{(k+3)!}$$

$$= 1 - \frac{2^{k+2}}{(k+3)!}$$

Since true for $n=1$, and true for $n=k$

$\Rightarrow$ true for $n=k+1$, true $\forall k \in \mathbb{N}$

2

Find the Maclaurin expansion, upto and including the term in x^4 of

$$\ln(\cos x)$$

$f(x) = \ln(\cos x) \qquad f(0) = 0$

$f'(x) = \dfrac{-\sin x}{\cos x} = -\tan x \qquad f'(0) = 0$

$f''(x) = -\sec^2 x \qquad f''(0) = -1$

$f'''(x) = -2\sec x \sec x \tan x \qquad f'''(0) = 0$

$= -2\sec^2 x \tan x$

$f^{(4)}(x) = -2\sec^2 x \sec^2 x - 2\sec^2 x \tan x \tan x$

$f^{(4)}(0) = -2$

$= -2\sec^4 x - 2\sec^2 x \tan^2 x$

$$0 + 0x - \frac{1x^2}{2!} + 0x^3 - \frac{2x^4}{4!}$$

$$\Rightarrow -\frac{1x^2}{2!} - \frac{x^4}{12}$$

3

(a) Expand $\left(z+\frac{1}{z}\right)^4$

(b) Hence, by considering $\left(z+\frac{1}{z}\right)^4$ and $\left(z-\frac{1}{z}\right)^4$, with $z=\cos\theta+i\sin\theta$ show that

$$\cos^4\theta+\sin^4\theta=\frac{1}{4}(\cos 4\theta+3)$$

a) $z^4 + 4z\left(\frac{1}{z}\right)^3 + 6z^2\left(\frac{1}{z}\right)^2 + 4z^3\left(\frac{1}{z}\right) + \frac{1}{z^4}$

$= z^4 + \frac{4}{z^2} + 6 + 4z^2 + \frac{1}{z^4}$

b) $z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2}\right) + 6 = \left(z+\frac{1}{z}\right)^4$

$z^4 + \frac{1}{z^4} - 4\left(z^2 + \frac{1}{z^2}\right) + 6 = \left(z-\frac{1}{z}\right)^4$

$z + \frac{1}{z} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$= 2\cos\theta$

$z - \frac{1}{z} = 2i\sin\theta$

$\left(z+\frac{1}{z}\right)^4 + \left(z-\frac{1}{z}\right)^4 = 16\cos^4\theta + 16\sin^4\theta$

$2z^4 + \frac{2}{z^4} + 12 = 16\left(\cos^4\theta + \sin^4\theta\right)$

$4(\cos 4\theta) + 12 = 16\left(\cos^4\theta + \sin^4\theta\right)$

$\frac{1}{4}(\cos 4\theta + 3) = \cos^4\theta + \sin^4\theta$

4

By means of the substitution $y = vx$ reduce the differential equaton

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + \sqrt{x^2 + y^2}$$

to an equation in v and x.
Find the solution , given that $y = 1$ when $x = 1$ in the form $y^2 = f(x)$

$y = vx$

$\frac{dy}{dx} = v + \frac{dv}{dx} x$

$x(vx)\left(v + \frac{dv}{dx}x\right) = v^2x^2 + \sqrt{x^2 + v^2x^2}$

$x^2v^2 + x^3v\frac{dv}{dx} = v^2x^2 + x\sqrt{1+v^2}$

$x^3v\frac{dv}{dx} = x\sqrt{1+v^2}$

$\int \frac{v}{\sqrt{1+v^2}} dv = \int \frac{1}{x^2} dx$

$(1+v^2)^{\frac{1}{2}} = -x^{-1} + c$

$y = (1+v^2)^{\frac{1}{2}}$

$y' = \frac{\frac{1}{2}2v}{(1+v^2)^{\frac{1}{2}}}$

$v^2 = \frac{y^2}{x^2}$

$\left(1 + \frac{y^2}{x^2}\right)^{\frac{1}{2}} = -\frac{1}{x} + c$

$(1+1)^{\frac{1}{2}} = -\frac{1}{1} + c$

$2^{\frac{1}{2}} = -1 + c$

$c = 2^{\frac{1}{2}} + 1$

$\left(1 + \frac{y^2}{x^2}\right)^{\frac{1}{2}} = -\frac{1}{x} + \sqrt{2} + 1$

$1 + \frac{y^2}{x^2} = \left(-\frac{1}{x} + \sqrt{2} + 1\right)^2$

$1 + \frac{y^2}{x^2} = \frac{1}{x^2} - \frac{2(\sqrt{2}+1)}{x} + 2\sqrt{2} + 3$

$\frac{y^2}{x^2} = \frac{1}{x^2} - \frac{2x(\sqrt{2}+1)}{x^2} + 2\sqrt{2} + 2$

$y^2 = 1 - 2(\sqrt{2}+1)x + 2(\sqrt{2}+1)x^2$

5

Prove that

$$\frac{\sin\theta}{1-\cos\theta} \equiv \cot\frac{1}{2}\theta$$

Let $t = \tan\frac{\theta}{2}$

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\frac{2t}{1+t^2} \div \left(1 - \frac{1-t^2}{1+t^2}\right)$$

$$= \frac{2t}{1+t^2} \div \left(\frac{1+t^2-1+t^2}{1+t^2}\right)$$

$$= \frac{2t}{1+t^2} \div \left(\frac{2t^2}{1+t^2}\right)$$

$$= \frac{2t}{2t^2}$$

$$= \frac{1}{t} = \underline{\underline{\cot\tfrac{1}{2}\theta}} \quad \square$$